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Decomposing a simple polygon into pseudo-triangles and convex polygons ☆

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Abstract

In this paper we consider the problem of decomposing a simple polygon into subpolygons that exclusively use vertices of the given polygon. We allow two types of subpolygons: pseudo-triangles and convex polygons. We call the resulting decomposition *PT-convex*. We are interested in *minimum* decompositions, i.e., in decomposing the input polygon into the least number of subpolygons. Allowing subpolygons of one of two types has the potential to reduce the complexity of the resulting decomposition considerably.

The problem of decomposing a simple polygon into the least number of *convex* polygons has been considered. We extend a dynamic-programming algorithm of Keil and Snoeyink for that problem to the case that both convex polygons and pseudo-triangles are allowed. Our algorithm determines such a decomposition in $O(n^3)$ time and space, where n is the number of the vertices of the polygon.
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1. Introduction

The problem of decomposing the convex hull of a set of points into subpolygons has a long history. We are interested in decompositions where the vertices of the subpolygons are restricted to the set of input points. Triangulations are an example of such decompositions. Every triangulation of a set of n points consists of $2n - 2 - c$ triangles, where c is the number of points on the convex hull. Thus one is usually interested in triangulations that optimize some additional parameter. For example, the Delaunay triangulation is known to maximize the smallest angle over all triangles in the triangulation. Another famous example is the minimum-weight triangulation. It minimizes the sum over the lengths of all edges in the triangulation. The complexity of computing this triangulation was open for a long time, until Mulzer and Rote [15] very recently managed to show that the problem is NP-hard. Krznaric and Levcopoulos [14]

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have shown that there is a constant-factor approximation for the problem, but the factor of their approximation is so large that they have not explicitly calculated it.

The concept of triangulations has been generalized by considering decompositions that consist of other, more complex and thus potentially fewer subpolygons. Two natural generalizations of triangles are (a) convex polygons and (b) pseudo-triangles. *Pseudo-triangles* are simple polygons with exactly three convex angles, i.e., interior angles of less than 180°. They have applications in visibility complexes [16], ray shooting [5,8], rigidity theory and robot arm motion planning [19], guarding polygons [17], and kinetic collision detection [12].

Concerning the first generalization of triangles, Fevens et al. [6] have investigated minimum convex decompositions, i.e., decompositions that consist of the least number of convex subpolygons. Their algorithm takes $O(n^{3h+3})$ time, where h is the number of nested convex hulls of the given point set. Spillner [18] has given a fixed-parameter algorithm for the problem, the number of points in the interior of the convex hull being the parameter. For the case that points are in general position, Knauer and Spillner [13] have given a simple 3-approximation that runs in $O(n \log n)$ time and a more involved 30/11-approximation that runs in $O(n^2)$ time.

Concerning the second generalization of triangles, Streinu [19] has shown strong links between minimally rigid graphs and minimum pseudo-triangulations. In particular, she proved that the minimum number of edges needed to obtain a pseudo-triangulation is $2n - 3$ and thus, by Euler's polyhedron theorem, the number of pseudo-triangles in a minimum pseudo-triangulations is $n - 2$, which does not depend on the structure of the point set (given general position) but only on its size. There has also been work on enumerating all minimum pseudo-triangulations [2,3]. Gudmundsson and Levcopoulos [9] investigate the problem of computing minimum-*weight* pseudo-triangulations for sets of points, where the weight of a decomposition is the sum over the lengths of all edges in the decomposition. They approximate the problem in two ways. Given a set of n points, their first algorithm computes a pseudo-triangulation whose weight is at most $O(\log n)$ times larger than that of a minimum spanning tree of the same point set. In contrast, they show there are point sets where every convex decompositions (and thus every triangulation) has weight $\Omega(n)$ times that of a minimum spanning tree. Their second algorithm computes in cubic time a pseudo-triangulation whose weight is at most 15 times that of a minimum-weight pseudo-triangulation.

Aichholzer et al. [1] were the first to investigate decompositions where each subpolygon has the choice to be convex or a pseudo-triangle, i.e., one of the two generalizations of triangles mentioned above. We call such decompositions *PT-convex*. They show that each minimum PT-convex decomposition of a set of n points consists of less than $7n/10$ polygons. In contrast, there are point sets where any minimum *convex* decomposition consists of at least $12n/11 - 2$ subpolygons [7]. (On the other hand, Knauer and Spillner [13] showed that every point set can be decomposed into no more than $15n/11 - 24/11$ convex polygons.)

A related problem is the decomposition of *simple polygons* into convex polygons or pseudo-triangles, e.g., for point location or ray shooting. When decomposing a simple polygon we also say that the decomposition is convex, a pseudo-triangulation or PT-convex if the decomposition uses exclusively the corresponding types of polygons. Again we are interested in minimum decompositions. Keil [10] has given a general technique for decomposing a simple polygon into polygons of a certain type. The technique is based on optimally decomposing subpolygons each of which is obtained from the original by drawing a single diagonal. Keil's technique yields an $O(nr^2 \log r)$-time algorithm for the convex decomposition problem, where r is the number of *reflex* vertices of the polygon, i.e., vertices whose inner angle is larger than 180°. Keil also showed that the convex decomposition problem becomes NP-hard if the input polygons can have holes. Keil and Snoeyink [11] improve Keil's algorithm by giving a $O(n + \min(nr^2, r^4))$-time and -space solution. Interestingly, the problem can be solved faster, namely in $O(n + r^3)$ time, when allowing Steiner points [4].

In the above-mentioned paper [9], Gudmundsson and Levcopoulos also give a cubic-time algorithm for computing a minimum-weight pseudo-triangulation of a simple polygon. They use this algorithm as a sub-routine for their 15-approximation of the minimum-weight pseudo-triangulation of a set of points.

In this paper we give an algorithm for computing a minimum PT-convex decomposition of a simple polygons. Our dynamic-programming algorithm is based on two main ingredients: Keil's general decomposition technique [10] and the way how Gudmundsson and Levcopoulos [9] determine all geodesics in the given polygon which form chains of reflex vertices and can thus potentially be sides of pseudo-triangles. Our algorithm takes $O(n^3)$ time and space.

Our paper is structured as follows. We first briefly describe the approach of Keil [10] and Keil and Snoeyink [11] in Section 2. Then we characterize pseudo-triangles in terms of chains of reflex vertices and vice versa, see Section 3.

We present our algorithm in Section 4 and analyze its running time in Section 5. Finally, we give some open problems in Section 6.

2. Previous work

Keil [10] introduces a general technique for decomposing a simple polygon into polygons of a certain type. The technique is based on optimally decomposing subpolygons each of which is obtained from the original by drawing a single diagonal d. In each decomposition $\mathcal{D}$ of a subpolygon there is a unique polygon $P(\mathcal{D})$ that contains the diagonal.

Keil defines a relation $D_1 \leqslant D_2$ between two minimum decompositions of a subpolygon if and only if the angles at d in the polygon $P(D_1)$ are not greater than the corresponding angles in $P(D_2)$ respectively. He argues that it suffices to consider decompositions that are minimal under this relation in order to find a minimum decomposition of P. Keil considers the equivalence classes which these minimal elements determine. Their representatives can be easily computed and can be used to check whether a given minimum decomposition can be extended without increasing the number of polygons. This idea yields an $O(n^3 \log n)$-time algorithm for the convex decomposition problem. Keil and Snoeyink [11] observe that (a) once a representative cannot be used to extend a decomposition it can be discarded and that (b) only diagonals incident on at least one reflex vertex need to be considered. This results in an $O(nr^2)$-time algorithm, where r is the number of reflex vertices of the polygon. Observation (a) helps us to obtain an $O(n^3)$-time algorithm for our problem.

3. Characterization of pseudo-triangles

We use $P^+(A_i, A_j)$ and $P^-(A_i, A_j)$ to denote the paths on the boundary ∂P from a vertex A_i to a vertex A_j of P in clockwise and counter-clockwise direction, respectively. To simplify the notation we will always assume that the edge $A_n A_1$ does not lie in the part of the polygon that we currently investigate; for three vertices A_i, A_j, and A_k, this means that we can write $i < j < k$ if we mean that vertex A_j lies on the path $P^+(A_i, A_k)$ (see Lemma 1, for an example).

We say that a point $Q \in P$ is *visible* from a point $Q' \in P$ if the relative interior of the line segment QQ' is contained in the interior of P or if Q and Q' are adjacent vertices of P; note that the *relative interior* of a line segment is the set of all the points of the segment except for the endpoints. With $\mathrm{vis}(A_i)$ we denote the list of all vertices of P that are visible from A_i in clockwise order starting with A_{i+1}. Unless stated otherwise, we number the vertices of a polygon in clockwise order.

Definition 1. Let $P = A_1 A_2 \ldots A_n$ be a simple polygon. A path $\Pi = B_1 B_2 \ldots B_m$ from $A_i = B_1$ to $A_j = B_m$ is a *concave geodesic* with respect to the polygon P if Π satisfies the following two conditions, see Fig. 1:

(G1) for each $k < m$ it holds that B_{k+1} is the last vertex on $P^+(B_k, A_j)$ which is visible from B_k, and
(G2) $B_1 B_2 \ldots B_m$ is a convex, counter-clockwise oriented polygon.

Remark 1. If $B_1 B_2 \ldots B_m$ is a concave geodesic from B_1 to B_m with respect to a simple polygon P and $m \geqslant 3$, then $B_2 \ldots B_m$ is a concave geodesic from B_2 to B_m with respect to P. On the other hand, given two vertices A_i and A_j

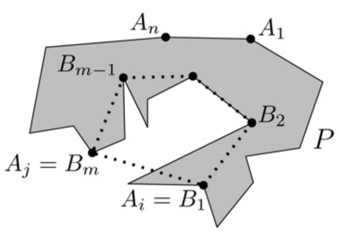


Fig. 1. The geodesic $B_1 B_2 \ldots B_m$ from A_i to A_j is concave with respect to the simple polygon P.

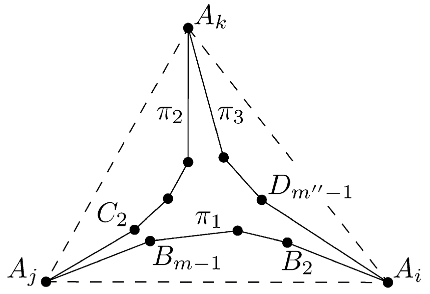


Fig. 2. Three consecutive concave geodesics π_1, π_2, and π_3 define a pseudo-triangle.

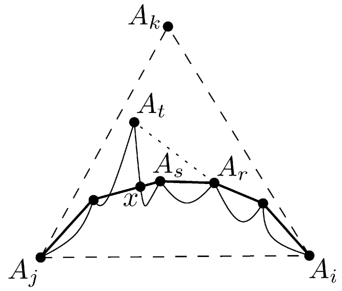


Fig. 3. Sketch for the proof of Lemma 1; arcs represent the boundary of the polygon $P^+(A_i, A_j)$, solid straight-line segments are the edges of $\pi_1 = T^+(A_i, A_j)$.

of a simple polygon P, there is at most one concave geodesic that connects A_i and A_j. Thus there are $O(n^2)$ concave geodesics in a simple polygon with n vertices.

For our further considerations we need the following observation.

Observation 1. *(See [11].) Let A_i be a vertex of $P = A_1A_2 \dots A_n$. Consider the line segments A_iA_j with $A_j \in$ $\text{vis}(A_i)$. Their cyclic order around A_i is the same as the order of their other endpoints along ∂P.*

We now state the relationship between pseudo-triangles and concave geodesics in a simple polygon.

Lemma 1. *Let $P = A_1A_2 \dots A_n$ be a simple polygon and let $T \subseteq P$ be a pseudo-triangle whose vertices are vertices of P and whose convex vertices are A_i, A_j and A_k with $i < j < k$. Then the paths $\pi_1 = T^+(A_i, A_j)$, $\pi_2 = T^+(A_j, A_k)$, and $\pi_3 = T^+(A_k, A_i)$ are concave geodesics with respect to P.*

Proof. First note that the vertices of π_1 lie on $P^+(A_i, A_j)$, those of π_2 lie on $P^+(A_j, A_k)$, and those of π_3 lie on $P^+(A_k, A_i)$ otherwise T would not be simple. To avoid double indices, let $B_1B_2 \dots B_m = \pi_1$, $C_1C_2 \dots, C_{m'} = \pi_2$, and $D_1D_2 \dots, D_{m''} = \pi_3$, see Fig. 2. We now check the two properties of concave geodesics for π_1. Again, to simplify the notation, we assume that π_1 does not contain the edge A_nA_1. The proofs for π_2 and π_3 are symmetric.

Now we establish property (G1). We consider the path π_1 and show that the construction proposed in property (G1) does not fail. Assume to the contrary that there is an index $r \in \{i, \dots, j-1\}$ such that A_r violates the construction proposed in property (G1). See Fig. 3. Let A_s be the vertex on π_1 immediately following A_r. Then it is clear that $s > r$. Now let $t \in \{s+1, \dots, j\}$ be such that A_t is visible from A_r. Due to Observation 1 we know that the edges A_rA_{r+1}, A_rA_s and A_rA_t appear in clockwise order around A_r. In particular, because of the convexity of $T^+(A_r, A_j)A_r$, the edge A_rA_t intersects $T^+(A_r, A_j)$ only in A_r and the edge A_rA_s is contained in the polygon $P^+(A_r, A_t)A_r$. However, A_j lies outside this polygon and thus $T^+(A_r, A_j)$ leaves $P^+(A_r, A_t)A_r$ in some point x which does not belong to A_rA_t, see Fig. 3. Hence $T^+(A_r, A_j)$ leaves P. This contradicts the fact that T is contained in P. Thus the assumption that property (G1) fails at some point is wrong. This shows that π_1 actually does satisfy property (G1).

Finally we show that property (G2), i.e., convexity, holds for the polygon $B = B_1B_2 \dots B_{m'}$. Consider the ray that emanates from A_i and goes through B_2. Note that the line segment A_iB_2 lies in P since T is contained in P. Due to Observation 1 we know that if we turn the ray in clockwise direction, it will hit $D_{m''-1}$. During this movement the part of the ray in a small neighborhood of A_i will remain in P. Denote by r the ray in an arbitrary position during this movement. Define an analogous ray r' emanating from A_j. Since the chain π_1 has reflex angles at $B_2, \dots, B_{m-1}$ in the interior of T (which correspond to convex angles in the interior of B), the chain cannot leave the triangle Δ formed by the line segment A_iA_j and the two rays r and r'. Since Δ has convex angles in $A_i = B_1$ and $A_j = B_{m'}$, this also holds for B. Thus all angles in B are convex, and so is B. $\square$

Next we establish the converse relation: three concave geodesics determine a pseudo-triangle.

Lemma 2. *Let $P = A_1A_2 \dots A_n$ be a simple polygon. Further let $i < j < k$ and $\pi_1 = A_i \dots A_j$, $\pi_2 = A_j \dots A_k$ and $\pi_3 = A_k \dots A_i$ be concave geodesics with respect to P. Then $\pi_1\pi_2\pi_3$ is a pseudo-triangle.*

Proof. To avoid double indices let $\pi_1 = A_i \dots A_j = B_1B_2 \dots B_m$ and $\pi_2 = A_j \dots A_k = C_1C_2 \dots C_{m'}$, see Fig. 2. We first consider the geodesics π_1 and π_2 and show that they are disjoint except for the vertex A_j where π_1 meets π_2 in a convex angle.

Rotate a ray counter-clockwise around A_j starting at A_i. Due to property (G2) in Definition 1 the ray sweeps over the vertices $A_i = B_1, \dots, B_{m-1}$ of π_1 in this order. Due to Observation 1 the ray then hits all vertices of $P^-(A_i, A_k)$ visible from A_j. Due to property (G1) the next vertex hit by the ray is C_2. Again due to (G2) the ray then hits the vertices $C_3, \dots, C_{m'} = A_k$ of π_2 in this order. This shows that π_1 and π_2 do not intersect and that the angle $\angle B_{m-1}A_jC_2$ is convex. Symmetric arguments show that analogous statements hold for the other two pairs (π_2, π_3) and (π_3, π_1) of geodesics. Due to property (G2) all vertices of $\pi_1\pi_2\pi_3$ other than A_i, A_j, and A_k are reflex. Thus $\pi_1\pi_2\pi_3$ is a simple polygon with exactly three convex vertices, i.e., a pseudo-triangle. □

4. Algorithm

We use the same approach for finding a minimum PT-convex decomposition of a simple polygon as Keil and Snoeyink [11] use for finding the minimum *convex* decomposition of a polygon, i.e., we consider subpolygons which are obtained from the original polygon by drawing a single diagonal. As Keil and Snoeyink we use dynamic programming, treating subpolygons in order of increasing number of vertices. For each subpolygon P' of the given polygon P we consider the diagonal d that separates P' from P. We compute two decompositions of P', namely one that is minimum under the constraint that d bounds a convex face of P' and one that is minimum given that d bounds a pseudo-triangle in P'.

We compute the smallest decomposition of the first type analogously to Keil and Snoeyink. For the second type of decomposition where the diagonal d bounds a pseudo-triangle we proceed as follows. Assume we have a precomputed list L of all concave geodesics with respect to P. Then we can filter L to find all pseudo-triangles that contain the diagonal d as an edge. For each such pseudo-triangle T we compute the size of a minimum decomposition that contains T. Among all these decompositions we keep the smallest.

Now we describe our algorithm in detail. Let $P = A_1A_2 \dots A_n$ be a simple polygon. We use definitions similar to those of Keil and Snoeyink. If $i < j$ and A_j is visible from A_i in P then we denote the line segment A_iA_j by d_{ij} and call it a *diagonal* of P. Note that by our definition of visibility, each edge of P is a diagonal. Each diagonal defines a (simple) subpolygon $P_{ij} = A_iA_{i+1} \dots A_j$ contained in P. If the diagonal is an edge of P, the subpolygon is empty.

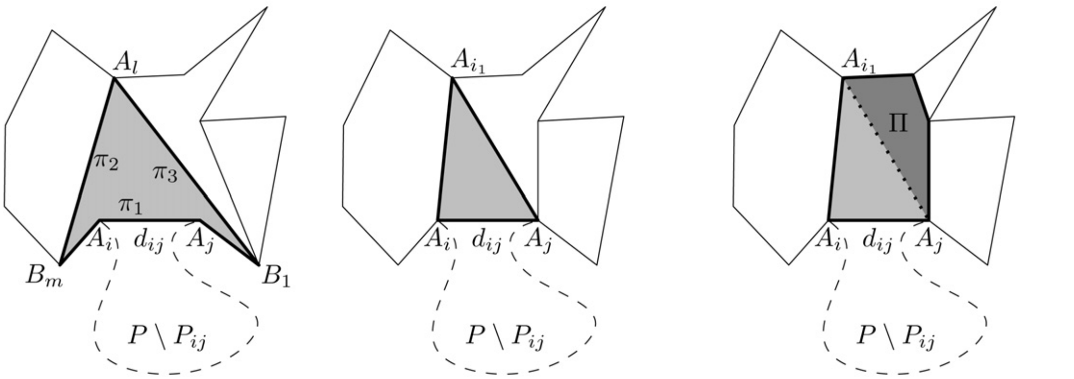
Definition 2. Let $\mathcal{D}_{ij}$ denote the set of all PT-convex decompositions of a polygon P_{ij} and let

$$\begin{aligned} w_{ij} &= \min\{|\mathcal{D}|:\ \mathcal{D} \in \mathcal{D}_{ij}\}, \\ cw_{ij} &= \min\{|\mathcal{D}|:\ \mathcal{D} \in \mathcal{D}_{ij} \text{ and the diagonal } d_{ij} \text{ is an edge of a convex polygon of } \mathcal{D}\}, \\ pw_{ij} &= \min\{|\mathcal{D}|:\ \mathcal{D} \in \mathcal{D}_{ij} \text{ and the diagonal } d_{ij} \text{ is an edge of a pseudo-triangle of } \mathcal{D}\}. \end{aligned}$$

Since the polygons $P_{i,i+1}$ $(i = 1, \dots, n)$ are degenerate we have that $w_{i,i+1} = cw_{i,i+1} = pw_{i,i+1} = 0$. Note also that $w_{ij} = \min(cw_{ij}, pw_{ij})$.

4.1. Computation of pw_{ij}

We describe how to find pw_{ij} given the values w_{kl} for each pair (k, l) with $(l - k) \bmod n < (j - i) \bmod n$ and a list L of all concave geodesics for the polygon P. We consider all concave geodesics which contain the edge A_iA_j and lie completely in P_{ij}. For each such geodesic $\pi_1 = B_1B_2 \dots B_m$ we go along $P^-(B_1, B_m)$ and for each vertex $A_l \in P^-(B_1, B_m)$ we check whether there exist concave geodesics $\pi_2 = B_m \dots A_l$ and $\pi_3 = A_l \dots B_1$. If π_2 and π_3 exist, then $\pi_1\pi_2\pi_3$ is a pseudo-triangle according to Lemma 2, see Fig. 4a. A minimum decomposition of P_{ij} containing this pseudo-triangle can be obtained if and only if for each pair $(k, l) \neq (i, j)$ such that A_kA_l is an edge of $\pi_1\pi_2\pi_3$ the polygon P_{kl} is optimally decomposed.



(a) d_{ij} bounds a pseudo-triangle (b) d_{ij} bounds the triangle $A_i A_{i_1} A_j$ (c) d_{ij} bounds a convex polygon $\Pi \cup A_i A_{i_1} A_j$

Fig. 4. Three types of minimum PT-convex decompositions of P_{ij}.

Thus if $w(\pi)$ denotes the sum of all w_{kl} where $A_k A_l$ lies on a geodesic π, then it is clear that the minimum decomposition of P_{ij} using the pseudo-triangle $\pi_1\pi_2\pi_3$ consists of

$$s(\pi_1, A_l) = 1 + w(\pi_2) + w(\pi_3) + \sum_{A_k A_l \in \pi_1 \setminus \{A_i A_j\}} w_{kl}$$

polygons. Now pw_{ij} is the minimum of $s(\pi_1, A_l)$ over all pairs (π_1, A_l) that fulfill the above requirements.

4.2. Computation of cw_{ij}

In this section we describe how to compute cw_{ij} if we have the values cw_{kl} and w_{kl} whenever $(l-k) \bmod n < (j-i) \bmod n$. Our approach is based on the algorithm of Keil and Snoeyink [11] for computing a minimum convex decomposition of a simple polygon. We start with the following definition.

Definition 3. A PT-convex decomposition $\mathcal{D}$ of P_{ij} is called *diagonal-convex* if the diagonal d_{ij} is an edge of a convex polygon $A_i A_{i_1} \ldots A_{i_m} A_j$ in $\mathcal{D}$, where $m > 1$.

Given a diagonal-convex decomposition $\mathcal{D}$ of P_{ij} that contains a convex polygon $A_i A_{i_1} \ldots A_{i_m} A_j$ it is clear that the triangle $A_i A_{i_1} A_j$ is contained in P. Moreover, if $m > 1$, then $A_{i_1} A_{i_2} \ldots A_j$ is also a (non-degenerate) convex polygon and thus $\mathcal{D}$ induces a diagonal-convex decomposition of $P_{i_1 j}$.

Now let us change the point of view. Consider a triangle $A_i A_{i_1} A_j$ with $i < i_1 < j$ that is contained in P. Then decomposing P_{ii_1} and $P_{i_1 j}$ optimally and adding the triangle $A_i A_{i_1} A_j$ yields a diagonal-convex decomposition of P_{ij}, see Fig. 4b. This decomposition $\mathcal{D}^\star$ consists of

$$w_{ii_1} + w_{i_1 j} + 1$$

polygons. Now the question is whether we can do better. Can we extend the triangle $A_i A_{i_1} A_j$ into a larger convex polygon $A_i A_{i_1} A_{i_2} \ldots A_j$? This is possible if and only if there is a diagonal-convex decomposition $\mathcal{D}'$ of $P_{i_1 j}$ containing a convex polygon $\Pi = A_{i_1} A_{i_2} \ldots A_{i_m} A_j$ with the additional property that

$$\angle A_{i_2} A_{i_1} A_i < 180^\circ \quad \text{and} \quad \angle A_i A_j A_{i_m} < 180^\circ. \tag{1}$$

Then we can merge the triangle $A_i A_{i_1} A_j$ with Π. This yields a diagonal-convex decomposition of P_{ij} consisting of $w_{ii_1} + |\mathcal{D}'|$ polygons. Note that if $\mathcal{D}'$ is not minimum, we have $w_{ii_1} + |\mathcal{D}'| \geqslant w_{ii_1} + cw_{i_1 j} + 1 \geqslant w_{ii_1} + w_{i_1 j} + 1 = |\mathcal{D}^\star|$, see Fig. 4c. Since we consider $\mathcal{D}^\star$, we can ignore non-minimum decompositions $\mathcal{D}'$. (Observe that there is no need to check whether the triangle $A_i A_{i_1} A_j$ can be extended by a convex polygon $A_i A_k \ldots A_{i_1}$ that is adjacent to edge $A_i A_{i_1}$; such a decomposition will be considered when we process the triangle $A_i A_k A_j$.)

It only remains to show how to test condition (1) efficiently. We do this similarly to Keil and Snoeyink [11]. First observe that in order to check condition (1) one only needs access to vertices A_{i_2} and A_{i_m} rather than to the

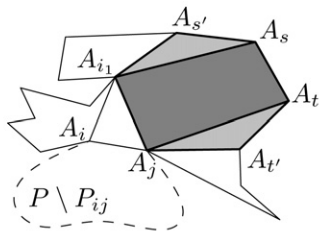


Fig. 5. The pair of indices (s,t) is a representative of $P_{i_1 j}$, while (s',t'), (s,s), and (t,t) are not.

whole polygon Π. Next consider two minimum diagonal-convex decompositions $\mathcal{D}$ and $\mathcal{D}'$ of $P_{i_1 j}$ with $\mathcal{D} \neq \mathcal{D}'$. Let $\Pi = A_{i_1} A_s \ldots A_t A_j$ and $\Pi' = A_{i_1} A_{s'} \ldots A_{t'} A_j$ be the corresponding convex polygons which contain the edge $A_{i_1} A_j$. Suppose that $s' \leqslant s \leqslant t \leqslant t'$. Then according to Observation 1, we have that

$$\angle A_i A_{i_1} A_s \leqslant \angle A_i A_{i_1} A_{s'} \quad \text{and} \quad \angle A_i A_j A_t \leqslant \angle A_i A_j A_{t'}.$$

Hence either $\mathcal{D}'$ violates condition (1), or if $\mathcal{D}'$ satisfies it, then $\mathcal{D}$ also satisfies it. For an example, see Fig. 5. In either case we can ignore $\mathcal{D}'$ without risking to lose the optimum solution. This is the motivation for defining a pair (s,t) of vertex indices with $i < s \leqslant t < j$ to be a *representative* for a polygon P_{ij} if the following two conditions hold.

(R1) The polygon P_{ij} has a minimum diagonal-convex decomposition that contains a convex polygon of the form $A_i A_s \ldots A_t A_j$.
(R2) For each other pair (s',t') satisfying condition (R1) it holds that $s' \leqslant s$ or $t \leqslant t'$.

We again refer to the example in Fig. 5. Condition (R2) yields the following observation.

Observation 2. *For each value of $s \in S = \{i+1, \ldots, j-1\}$ the subpolygon P_{ij} has at most one representative (s, t_s). Thus P_{ij} has at most $|S| \in O(n)$ representatives.*

Given the representatives of all P_{lj} with $i < l < j$ we can compute cw_{ij} as

$$cw_{ij} = \min\Big\{ \min_{\substack{i<l<j:\\ A_i A_l A_j \subset P}} (w_{il} + w_{lj} + 1),\ \min_{l}(w_{il} + cw_{lj}) \Big\}$$

where the last minimum is over all $i < l < j$ such that $A_i A_l A_j \subset P$ and additionally P_{lj} possesses a representative satisfying condition (1). The information about the representatives of all $P_{i_1 j}$ with $i < i_1 < j$ suffices to determine the representatives of P_{ij}. As Keil and Snoeyink [11] we maintain a list of representatives for each P_{ij} sorted with respect to the first component which allows us to compute all values of type cw_{ij} in amortized $O(n)$ time per pair (i,j).

5. Analysis

We now investigate the time and space complexity of our algorithm. In order to compute the values of type pw_{ij} we need access to all concave geodesics. The following slight modification of Theorem 2 in [9] yields a data structure that lets us compute and efficiently access the concave geodesics in a simple polygon. In order to motivate property (P3), observe that $A_{\min(\pi)}$ and $A_{\max(\pi)}$ are adjacent on a concave geodesic π if and only if π lies in the subpolygon $P_{\min(\pi)\max(\pi)}$ and contains the diagonal $d_{\min(\pi)\max(\pi)}$.

Proposition 1. *Given a simple polygon $P = A_1 A_2 \ldots A_n$ we can construct in $O(n^2)$ time and space a data structure DS with the following properties*:

(P1) *Given a pair (i,j), DS decides in $O(1)$ time whether there is a concave geodesic π from A_i to A_j.*
(P2) *If π is a concave geodesic, DS provides the minimum index* $\min(\pi) = \{l \mid A_l \in \pi\}$ *and the analogously defined maximum index* $\max(\pi)$ *of π in $O(1)$ time.*

(P3) *If π is a concave geodesic, DS decides in $O(1)$ time whether $A_{\min(\pi)}$ and $A_{\max(\pi)}$ are adjacent on π.*
(P4) *If π is a concave geodesic of length l, DS provides in $O(l)$ time a walk along π.*

Proof. We first compute all lists $\mathrm{vis}(A_i)$ in $O(n^2)$ total time. Then we use dynamic programming to check whether there is a concave geodesic π from A_i to A_j. If π exists, we also compute the second and the second last vertex on π. Having these vertices on each shorter geodesic, we can walk along π by repeatedly jumping to the second vertex of the remaining path, which by Remark 1 is also a geodesic.

We consider the pairs (i, j) in increasing order of the number of vertices on the path $P^+(A_i, A_j)$. The edges A_iA_{i+1} obviously correspond to concave geodesics and it is easy to determine the second and second last vertex of these paths.

When $P^+(A_i, A_j)$ consists of more than one edge, we use the list $\mathrm{vis}(A_i)$ to find the last vertex A_l visible from A_i on $P^+(A_i, A_j)$. Observation 1 allows us to extract the desired information from $\mathrm{vis}(A_i)$ in $O(1)$ amortized time. Then we query *DS* to see whether there is a concave geodesic π from A_l to A_j. (We can query *DS* with (l, j) since $(l - j) \bmod n < (i - j) \bmod n$.) If this is the case, we use the second and the second last vertex on π to check whether A_i can be added to π without violating property (G1). According to Remark 1 this is the only way for obtaining a concave geodesic π' from A_i to A_j. If A_j is visible from A_i, we can easily set $\min(\pi')$ and $\max(\pi')$, and determine the second and the second last vertex on π'. Moreover $A_{\min(\pi')}$ and $A_{\max(\pi')}$ are adjacent in this case.

Consider the more interesting case that A_j is not visible from A_i. Then the second vertex on π' is A_l and the second last vertex on π' is the second last vertex on π. It is also clear that $\min(\pi') = \min(i, \min(\pi))$ and $\max(\pi') = \max(i, \max(\pi))$. If $\min(\pi') \neq i$ and $\max(\pi') \neq i$, then $A_{\min(\pi')}$ and $A_{\max(\pi')}$ are adjacent on π' if and only if $A_{\min(\pi)}$ and $A_{\max(\pi)}$ are adjacent on π, a piece of information that we have already computed. Otherwise, say if $i = \min(\pi')$, then $A_{\min(\pi')}$ and $A_{\max(\pi')}$ are adjacent on π' if and only if $\max(\pi') = l$. Thus the information we earlier computed for π yields the desired information for π' in $O(1)$ time. It follows that we spend $O(1)$ (amortized) time and $O(1)$ space per pair (i, j), and $O(n^2)$ time and space in total. □

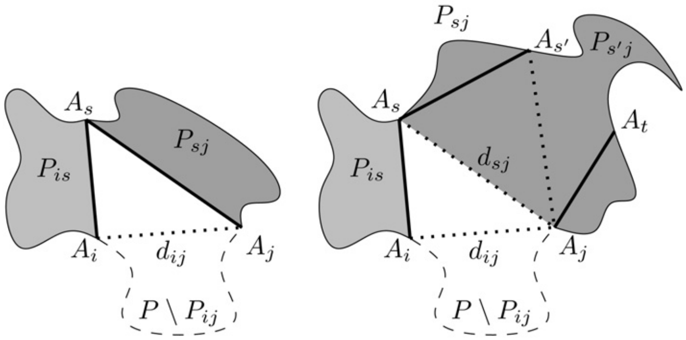
Now we are ready to prove our main theorem.

Theorem 2. *A minimum PT-convex decomposition of a simple polygon $P = A_1A_2 \ldots A_n$ can be computed in $O(n^3)$ time and space.*

Proof. Our algorithm for computing a minimum PT-convex decomposition is based on dynamic programming. We first detail how to compute the *number* of polygons in a minimum PT-convex decomposition rather than the decomposition itself.

We first set up the data structure *DS* of Proposition 1. Then for each subpolygon P_{ij} we compute a list L_{ij} consisting of all concave geodesics that are contained in P_{ij} and contain the diagonal d_{ij} as an edge. Recall that this is exactly the set of concave geodesics π with $\min(\pi) = i$ and $\max(\pi) = j$. We store $\pi = A_h, \ldots, A_i, A_j, \ldots, A_k$ simply as the pair (h, k) in L_{ij}. Note that each concave geodesic is contained in at most one list L_{ij}. We generate the lists of type L_{ij} as follows. We consider all pairs (i, j) in order of increasing value of $j - i$ and query *DS* to see whether there is a concave geodesic π from A_i to A_j. We then use *DS* to determine whether $A_{\min(\pi)}$ and $A_{\max(\pi)}$ are adjacent on π. If yes, we insert the information provided by *DS* for the concave geodesic π in the list $L_{\min(\pi)\max(\pi)}$. Since all the queries to *DS* so far require constant time each, the construction of the lists L_{ij} takes $O(n^2)$ time in total.

Then we implement the algorithm of Section 4. Using the technique of Keil and Snoeyink [11], we can compute all values of type cw_{ij} in $O(n^3)$ time in total. It remains to bound the time needed for computing the values of type pw_{ij}. We check each concave geodesic π in L_{ij}. (Recall that the lists of type L_{ij} are pairwise disjoint.) We walk along π to determine the sum of the values w_{kl} over all $(k, l) \neq (i, j)$ with $A_kA_l \subseteq \pi$. For each point A_l on $P^+(B_m, B_1)$ we check whether there is a concave geodesic π_1 from A_l to B_1 and a concave geodesic π_2 from B_m to A_l. If this is the case, then $\pi\pi_1\pi_2$ is a pseudo-triangle according to Lemma 2. In order to compute a minimum decomposition containing $\pi\pi_1\pi_2$, we need the values $w(\pi_1)$ and $w(\pi_2)$, which can be computed by walking along π_1 and π_2 the first time we need these values. In total, we walk along each geodesic a constant number of times. By Proposition 1, each walk takes $O(n)$ time. According to Remark 1, the total number of concave geodesics is $O(n^2)$. This implies that we can determine all values of type pw_{ij} in $O(n^3)$ time.



(a) when $s = t$ (b) when $s \neq t$

Fig. 6. Constructing a minimum decomposition of P_{ij}.

Thus the number of polygons in a minimum PT-convex decomposition of a simple polygon P with n vertices can be computed in $O(n^3)$ time. The algorithm Keil and Snoeyink [11] requires $O(n^3)$ space, while the data structure *DS* as well as the lists of type L_{ij} use $O(n^2)$ space.

Finally we show how to adjust the above dynamic program so that it also yields a minimum pseudo-convex decomposition. Whenever in that program we compute a value of type pw_{ij} or cw_{ij}, we do some additional bookkeeping. From this extra information we can then backtrack and compute the decomposition that corresponds to the result of the dynamic program. This is a usual trick in dynamic programming. In the backtracking phase, we initially set $i = 1$ and $j = n$, and then repeatedly check whether $w_{ij} = pw_{ij}$ or $w_{ij} = cw_{ij}$. The following two paragraphs describe what we do in the first and in the second case, respectively.

Recall that when computing p_{ij}, we determine the pseudo-triangle T that (a) lies in P_{ij}, (b) is adjacent to d_{ij}, and (c) yields the minimum decomposition of P_{ij} among all pseudo-triangles fulfilling (a) and (b). Now the above-mentioned extra bookkeeping consists of storing the indices of the three convex corners of T. When backtracking we draw the edges of T (with the help of *DS*) and further decompose the subpolygons that constitute $P_{ij} \setminus T$.

When computing cw_{ij}, we store a copy of the list of representatives of P_{ij}. Note that this list has been built completely by the time that cw_{ij} is computed. Take any representative (s, t) and draw the corresponding diagonals d_{is} and d_{tj}. We consider two cases. If $s = t$, we simply have to find minimum pseudo-convex decompositions of P_{is} and of P_{sj}, see Fig. 6a. Otherwise, if $s \neq t$, we decompose P_{is} into w_{is} polygons without any restrictions and decompose P_{sj} into cw_{sj} polygons including a convex polygon $A_s \ldots A_t A_j$, see Fig. 6b. When decomposing P_{sj} we must respect the diagonal d_{tj}. This is where we need the list of *all* representatives: by construction P_{sj} must have a representative (s', t), and we can recursively decompose $P_{s'j}$.

Let us analyze time and space consumption of the modified dynamic program. The total time we spend for the backtracking part is the number of diagonals we draw plus the time we spend scanning the representative lists. The diagonals (including the edges of P) are the edges of a plane graph with n vertices, thus their number is linear in n. Due to Observation 2 and Remark 1 the total number of representatives is $O(n^3)$. Storing (and scanning) these dominates the time and the space consumption of the modified dynamic program. □

6. Open problems

We have given an efficient and relatively simple dynamic program for computing minimum PT-convex decompositions of simple polygons. Can the running time of $O(n^3)$ be improved, e.g., by making it not only depend on the number n of vertices, but also on, say, the number r of reflex vertices?

Can minimum PT-convex decompositions for *point sets* be computed efficiently? Is there a way to get at least a constant-factor approximation for that problem by decomposing the convex hull of the point set into simple polygons and then using our algorithm to further decompose the simple polygons? Note that grouping the subpolygons of a minimum decompositions arbitrarily into simple polygons and decomposing these simple polygons one after the other with our algorithm will yield a minimum decomposition, possibly different from the original. This shows that there are always decompositions of the point set into simple polygons that yield minimum PT-convex decompositions. The problem is just to guess a "right" decomposition. Note that Gudmundsson and Levcopoulos [9] use a similar strategy in their 15-approximation of the minimum-*weight* pseudo-triangulation.

Finally we ask whether PT-convex decompositions with other optimality criteria can be computed efficiently, e.g., minimum total edge weight. It seems that the method of representatives [11] will fail here since it is tailored towards minimizing the *number* of subpolygons.

One of the anonymous referees of this paper suggested to allow Steiner points to get even smaller PT-convex decompositions. This is an interesting variant of the problem. Recall that the algorithm of Chazelle and Dobkin [4] that computes minimum convex decompositions with Steiner points is faster than the fastest known algorithm for the corresponding problem without Steiner points [11].

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References

[1] O. Aichholzer, C. Huemer, S. Kappes, B. Speckmann, C.D. Tóth, Decompositions, partitions, and coverings with convex polygons and pseudo-triangles, Graph. Comb. 23 (5) (2007) 481–507.
[2] O. Aichholzer, G. Rote, B. Speckmann, I. Streinu, The zigzag path of a pseudo-triangulation, in: F. Dehne, R. Sack, M. Smid (Eds.), Proc. 8th Workshop on Algorithms and Data Structures (WADS'03), in: Lecture Notes Comput. Sci., vol. 2748, Springer-Verlag, 2003, pp. 377–388.
[3] S. Bereg, Enumerating pseudo-triangulations in the plane, Comput. Geom. Theory Appl. 30 (3) (2005) 207–222.
[4] B. Chazelle, D.P. Dobkin, Optimal convex decompositions, in: G.T. Toussaint (Ed.), Computational Geometry, North Holland, Amsterdam, 1985, pp. 63–133.
[5] B. Chazelle, H. Edelsbrunner, M. Grigni, L.J. Guibas, J. Hershberger, M. Sharir, J. Snoeyink, Ray shooting in polygons using geodesic triangulations, Algorithmica 12 (1) (1994) 54–68.
[6] T. Fevens, H. Meijer, D. Rappaport, Minimum convex partition of a constrained point set, Discrete Appl. Math. 109 (1–2) (2001) 95–107.
[7] J. García-López, C.M. Nicolás, Planar point sets with large minimum convex partitions, in: Proc. 22nd European Workshop Comput. Geom. (EWCG'06), Delphi, 2006, pp. 51–54.
[8] M.T. Goodrich, R. Tamassia, Dynamic ray shooting and shortest paths in planar subdivisions via balanced geodesic triangulations, J. Algorithms 23 (1) (1997) 51–73.
[9] J. Gudmundsson, C. Levcopoulos, Minimum weight pseudo-triangulations, Comput. Geom. Theory Appl. 38 (3) (2007) 139–153.
[10] J.M. Keil, Decomposing a polygon into simpler components, SIAM J. Comput. 14 (1985) 799–817.
[11] J.M. Keil, J. Snoeyink, On the time bound for convex decomposition of simple polygons, Internat. J. Comput. Geom. Appl. 12 (3) (2002) 181–192.
[12] D.G. Kirkpatrick, J. Snoeyink, B. Speckmann, Kinetic collision detection for simple polygons, Internat. J. Comput. Geom. Appl. 12 (1–2) (2002) 3–27.
[13] C. Knauer, A. Spillner, Approximation algorithms for the minimum convex partition problem, in: Proc. 10th Scandinavian Workshop Algor. Theo. (SWAT'06), in: Lecture Notes Comput. Sci., vol. 4059, Springer-Verlag, 2006, pp. 232–241.
[14] D. Krznaric, C. Levcopoulos, Quasi-greedy triangulations approximating the minimum weight triangulation, J. Algorithms 27 (2) (1998) 303–338.
[15] W. Mulzer, G. Rote, Minimum weight triangulation is NP-hard, in: Proc. 22nd Annu. ACM Sympos. Comput. Geom. (SoCG'06), Sedona, Arizona, 2006, pp. 1–10.
[16] M. Pocchiola, G. Vegter, Topologically sweeping visibility complexes via pseudo-triangulations, Discrete Comput. Geom. 16 (1996) 419–453.
[17] B. Speckmann, C.D. Tóth, Allocating vertex π-guards in simple polygons via pseudo-triangulations, Discrete Comput. Geom. 33 (2) (2005) 345–364.
[18] A. Spillner, Optimal convex partitions of point sets with few inner points, in: Proc. 17th Canadian Conf. Comput. Geom. (CCCG'05), Windsor, 2005, pp. 34–37.
[19] I. Streinu, A combinatorial approach to planar non-colliding robot arm motion planning, in: Proc. 41st Annu. IEEE Sympos. Foundat. Comput. Sci. (FOCS'00), 2000, pp. 443–453.